J. Fluid Mech. (1969), vol. 35, part 2, pp. 243–254
Printed in Great Britain

Effect of modulation on the onset of thermal convection

By GIULIO VENEZIAN†
California Institute of Technology, Pasadena, California 91109

(Received 21 February 1968)

The stability of a horizontal layer of fluid heated from below is examined when, in addition to a steady temperature difference between the walls of the layer, a time-dependent sinusoidal perturbation is applied to the wall temperatures. Only infinitesimal disturbances are considered. The effects of the oscillating temperature field are treated by a perturbation expansion in powers of the amplitude of the applied field. The shift in the critical Rayleigh number is calculated as a function of frequency, and it is found that it is possible to advance or delay the onset of convection by time modulation of the wall temperatures.

1. Introduction

The effect of modulation on the stability of the flow between rotating cylinders was investigated experimentally by Donnelly (1964). In his experiments, fluid was confined in the narrow gap between two cylinders, with the outer cylinder held fixed while the inner cylinder was given an angular speed $\Omega + \Delta\Omega \cos \omega t$. He found that the onset of instability was delayed by the modulation of the angular speed of the inner cylinder. Maximum stability was achieved for $\omega d^2/\nu \simeq 0{\cdot}27$, and, as the frequency was increased far beyond that point, the effect of modulation became negligible. Donnelly interpreted his results as being due to a viscous wave penetrating the fluid and thereby altering the profile from an unstable one to a stable one.

Since the problems of Taylor stability and Bénard stability are very similar, and the latter is simpler to analyse, this paper deals with the thermal analogue of Donnelly's experiments. The problem considered is that of determining the onset of convection for a fluid layer heated from below, when, in addition to a fixed temperature difference between the walls, an additional perturbation is applied to the wall temperatures, varying sinusoidally in time.

2. Statement of the problem

The problem considered is the following. A fluid layer is confined between two infinite horizontal walls, a distance L apart. A vertical gravity force acts on the fluid. The wall temperatures are externally imposed, and they are

$$T_R + \tfrac{1}{2}\Delta T[1 + \epsilon \cos \omega t] \tag{1}$$

† Present address: University of Hawaii, Honolulu, Hawaii 96822.

at the lower wall ($z = 0$), and

$$T_R - \tfrac{1}{2}\Delta T[1 - \epsilon \cos(\omega t + \phi)] \tag{2}$$

at the upper wall ($z = L$). Here ϵ represents a small amplitude.

The fluid is supposed to be essentially incompressible, except in so far as its density may change by thermal expansion. For small departures from a reference temperature T_R, the density is given by

$$\rho = \rho_R[1 - \alpha(T - T_R)], \tag{3}$$

where α is the coefficient of thermal expansion. The thermal diffusivity κ and the kinematic viscosity ν of the fluid will be regarded as constant. As a mathematical model we use the Boussinesq approximation, for which Chandrasekhar (1961) is a convenient reference.

For simplicity, 'free–free' boundary conditions will be applied at the wall, instead of the more physical no-slip conditions. The free–free conditions are that the normal velocity is zero and the tangential stress is zero at the wall. They correspond to a rigid but slippery wall.

The object of the analysis is to determine the critical conditions under which convection can occur.

3. The hydrostatic configuration

A hydrostatic configuration is possible for this system, in which the isothermal surfaces (and hence the isosteric surfaces) are horizontal and therefore parallel to the equipotential surfaces of the vertical gravitational force. Under such conditions a vertical pressure gradient can balance the gravitational force, and the fluid is subject to no net force. The equations which determine the temperature and pressure fields in this case are

$$-\frac{\partial p_H}{\partial z} = \rho_H g, \tag{4}$$

and

$$\frac{\partial T_H}{\partial t} = \kappa \frac{\partial^2 T_H}{\partial z^2}. \tag{5}$$

Equations (3), (4) and (5), together with the boundary conditions (1) and (2), determined the hydrostatic fields $T_H(z,t)$, $\rho_H(z,t)$ and $p_H(z,t)$.

We shall only need the temperature field T_H, which, since (5) is linear, consists of the sum of a steady temperature field T_S and an oscillating part ϵT_1:

$$T_H = T_S(z) + \epsilon T_1(z,t), \tag{6}$$

where

$$T_S = T_R + \Delta T(L - 2z)/2L, \tag{7}$$

and

$$T_1 = \mathrm{Re}\{[a(\lambda)\, e^{\lambda z/L} + a(-\lambda) \exp[-\lambda z/L]\, e^{-i\omega t}]\}. \tag{8}$$

In (8),

$$\lambda = (1 - i)\left(\frac{\omega L^2}{2\kappa}\right)^{\frac{1}{2}}, \tag{9}$$

and

$$a(\lambda) = \frac{\Delta T}{2} \frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}. \tag{10}$$

4. Equations of motion

In the Boussinesq approximation, the equations of motion are

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}.\nabla\mathbf{v}+\frac{1}{\rho_R}\nabla(p-p_H) = \nu\nabla^2\mathbf{v}+g\alpha(T-T_H)\mathbf{k}, \tag{11}$$

$$\nabla.\mathbf{v} = 0, \tag{12}$$

and

$$\frac{\partial T}{\partial t}+\mathbf{v}.\nabla T = \kappa\nabla^2 T, \tag{13}$$

where $\mathbf{k}$ is the unit vector in the vertical direction, and $\mathbf{v} = (u, v, w)$ is the fluid velocity.

Let
$$\theta = T-T_H, \tag{14}$$

then, retaining only linear terms in $\mathbf{v}$ and θ, the equations of motion are

$$\frac{1}{\sigma}\frac{\partial \mathbf{v}'}{\partial t'}+\nabla' p' = \nabla'^2\mathbf{v}'+R\theta'\mathbf{k}, \tag{15}$$

$$\nabla'.\mathbf{v}' = 0, \tag{16}$$

and

$$\frac{\partial \theta'}{\partial t}+w'\frac{\partial T'_H}{\partial z'} = \nabla'^2\theta'. \tag{17}$$

Here, the variables have been non-dimensionalized as follows:

$$\mathbf{r}' = \mathbf{r}/L, \quad t' = \kappa t/L^2, \quad T' = T/\Delta T,$$
$$\mathbf{v}' = L\mathbf{v}/\kappa, \quad p' = Lp/\rho_R\kappa^2.$$

The two dimensionless groups which appear are the Prandtl number, $\sigma = \nu/\kappa$, and the Rayleigh number, $R = g\alpha\Delta T L^3/\kappa\nu$.

From this point on we shall drop the primes, with the understanding that, unless otherwise stated, the quantities are in their non-dimensional form.

The boundary conditions at $z = 0$ and $z = 1$ are

$$w = 0 \text{ (rigid wall)},$$

$$\frac{\partial^2 w}{\partial z^2} = 0 \text{ (slippery walls)},$$

and
$$\theta = 0 \text{ (externally fixed temperature)}.$$

We are interested in non-zero solutions to (15) to (17) subject to these boundary conditions.

It is convenient to express the entire problem in terms of w. This is accomplished by taking the curl of (15) twice. The z component of the resulting equation involves only w and θ:

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 w = R\nabla_1^2\theta, \tag{18}$$

where $$\nabla_1^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

Equations (17) and (18) can then be combined to obtain

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = -R\frac{\partial T_0}{\partial z}\nabla_1^2 w. \tag{19}$$

The boundary conditions can also be expressed in terms of w by making use of (18), which requires $\partial^4 w/\partial z^4 = 0$ if w and θ are zero. Thus (19) has to be solved subject to the homogeneous conditions

$$w = \partial^2 w/\partial z^2 = \partial^4 w/\partial z^4 = 0 \quad \text{at} \quad z = 0, 1. \tag{20}$$

The temperature gradient appearing in (19) can be obtained from the expressions derived in §3:

$$\frac{\partial T_0}{\partial z} = -1 + \epsilon \operatorname{Re}\{[A(\lambda)\, e^{\lambda z} + A(-\lambda)\, e^{-\lambda z}]\, e^{-i\omega t}\}, \tag{21}$$

$$= -1 + \epsilon f,$$

where $$A(\lambda) = \frac{\lambda}{2}\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}.$$

The horizontal dependence of w is factorable in this problem, and we shall study only solutions with a single wave number, α, such that

$$\nabla_1^2 w = -\alpha^2 w.$$

The dependence $\exp(i\boldsymbol{\alpha}.\mathbf{r})$ of w on the horizontal co-ordinates is to be understood throughout, even though, for the sake of conciseness of notation, the exponential factor will be left out.

5. Perturbation procedure

We seek the eigenfunctions w and eigenvalues R of (19) and (20) for a temperature profile that departs from the linear profile $\partial T_0/\partial z = -1$ by quantities of order ϵ. It follows that the eigenfunctions and eigenvalues which obtain in this problem differ from those associated with the standard Bénard problem by quantities of order ϵ. Accordingly, we seek an expansion of the form

$$\left.\begin{aligned} w &= w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots, \\ R &= R_0 + \epsilon R_1 + \epsilon^2 R_2 + \ldots. \end{aligned}\right\} \tag{22}$$

This type of expansion was first used in connexion with convection problems by Malkus & Veronis (1958) to consider effects of finite-amplitude convection. More recently, a similar expansion was used among others by Schlüter, Lortz & Busse (1965) to study the stability of finite-amplitude convection, and by Ingersoll (1966) to study the effect of superposing a shear flow. This expansion is in effect a generalization of Rayleigh's perturbation procedure.

If the expansions (22) are substituted into (19) and the powers of ϵ are separated, the resulting system of equations is

$$\left.\begin{aligned} Lw_0 &= 0, \\ Lw_1 &= R_1\nabla_1^2 w_0 - R_0 f\nabla_1^2 w_0, \\ Lw_2 &= R_1\nabla_1^2 w_1 + R_2\nabla_1^2 w_0 - R_0 f\nabla_1^2 w_1 - R_1 f\nabla_1^2 w_0, \end{aligned}\right\} \tag{23}$$

where
$$L = \left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 - R_0\nabla_1^2. \tag{24}$$

Each of the w_n is required to satisfy the boundary conditions (20).

The function w_0 which starts the whole process is a solution of the problem with $\epsilon = 0$, that is, the classical Bénard problem. The marginally stable solutions for that problem are
$$w_0^{(n)} = \sin n\pi z,$$

with corresponding eigenvalues
$$R_0^{(n)} = \frac{(n^2\pi^2 + \alpha^2)^3}{\alpha^2}.$$

For a fixed value of α the least eigenvalue is
$$R_0 = \frac{(\pi^2 + \alpha^2)^3}{\alpha^2}, \tag{25}$$

corresponding to
$$w_0 = \sin \pi z. \tag{26}$$

We shall use these as the starting-point of our solution.

The equation for w_1 then reads
$$Lw_1 = -R_1\alpha^2 \sin \pi z + R_0 \alpha^2 f \sin \pi z. \tag{27}$$

If this equation is to have a solution, the right-hand side must be orthogonal to the null space of the operator L. In effect, this solubility condition requires that the time-independent part of the right-hand side should be orthogonal to $\sin \pi z$. Since f varies sinusoidally in time, the only steady term is $-R_1\alpha^2 \sin \pi z$, so that R_1 is zero. Indeed, this could have been foreseen because R should be independent of the sign of ϵ, since changing the sign of ϵ merely corresponds to a shift in the time origin by half a period. Since such a shift does not change the problem of stability, it follows that all the odd coefficients $R_1, R_3, \ldots$ are zero.

Although (27) in principle can be solved in closed form, it is more convenient to expand the right-hand side in a Fourier series, and thus obtain an expression for w_1 by inverting the operation L term by term. For this, the expansion of $e^{\lambda z}$ in a Fourier series is needed. For subsequent steps in the problem, we require the expansion of $e^{\lambda z} \sin m\pi z$. It is easily determined that

$$\begin{aligned} g_{nm}(\lambda) &= 2\int_0^1 e^{\lambda z} \sin n\pi z \sin m\pi z \, dz, \\ &= -\frac{4nm\pi^2\lambda[1 + (-1)^{n+m+1} e^{\lambda}]}{[\lambda^2 + (n+m)^2\pi^2][\lambda^2 + (n-m)^2\pi^2]}, \end{aligned} \tag{28}$$

so that
$$e^{\lambda z}\sin m\pi z = \sum_{n=1}^{\infty} g_{nm}\sin n\pi z. \tag{29}$$

It is convenient to define

$$L(\omega, n) = \frac{\omega^2}{\sigma}(n^2\pi^2+\alpha^2)+i\omega\left(1+\frac{1}{\sigma}\right)(n^2\pi^2+\alpha^2)^2-(n^2\pi^2+\alpha^2)^3+(\pi^2+\alpha^2)^3. \tag{30}$$

It follows that
$$L\sin n\pi z\, e^{-i\omega t} = L(\omega, n)\sin n\pi z\, e^{-i\omega t} \tag{31}$$

(with the horizontal dependence on $e^{i\boldsymbol{\alpha}\cdot\mathbf{r}}$ understood).

Equation (27) now reads

$$Lw_1 = R_0\alpha^2 \mathrm{Re}\{\Sigma[A(\lambda)g_{n1}(\lambda)+A(-\lambda)g_{n1}(-\lambda)]\,e^{-i\omega t}\sin n\pi z\},$$

so that
$$w_1 = R_0\alpha^2\mathrm{Re}\left\{\Sigma\frac{B_n(\lambda)}{L(\omega,n)}e^{-i\omega t}\sin n\pi z\right\}, \tag{32}$$

where
$$B_n(\lambda) = A(\lambda)g_{n1}(\lambda)+A(-\lambda)g_{n1}(-\lambda). \tag{33}$$

A term proportional to $\sin\pi z$ (the solution to the homogeneous equation) could be added. However, this would merely amount to a renormalization of w, since all the terms proportional to $\sin\pi z$ could then be regrouped to define a new w_0, with corresponding new definitions for the other w_n's. For this reason, it is convenient to assume from the outset that w_0 is orthogonal to all the other w_n's.

The equation for w_2 is
$$Lw_2 = -R_2\alpha^2 w_0+R_0\alpha^2 f w_1. \tag{34}$$

We shall not require the solution of this equation, but merely use it to determine R_2, the first non-zero correction to R. The solubility condition requires that the steady part of the right-hand side should be orthogonal to $\sin\pi z$, and therefore

$$R_2 = 2R_0\int_0^1 \overline{fw_1}\sin\pi z\,dz, \tag{35}$$

where the bar denotes a time average. Now, from (27)

$$f\sin\pi z = \frac{1}{\alpha^2 R_0}Lw_1,$$

so that
$$\overline{fw_1}\sin\pi z = \frac{1}{\alpha^2 R_0}\overline{w_1 Lw_1}$$
$$= \alpha^2\frac{R_0}{2}\mathrm{Re}\left\{\Sigma\frac{B_n(\lambda)}{L(\omega,n)}\sin n\pi z\,\Sigma B_n^*(\lambda)\sin n\pi z\right\},$$

and finally
$$R_2 = \alpha^2\frac{R_0^2}{2}\mathrm{Re}\,\Sigma\frac{|B_n(\lambda)|^2}{L(\omega,n)}$$
$$= \frac{\alpha^2 R_0^2}{4}\sum_{n=1}^{\infty}\frac{|B_n(\lambda)|^2}{|L(\omega,n)|^2}[L(\omega,n)+L^*(\omega,n)]. \tag{36}$$

Equation (34) could now be solved for w_2 if desired, and the procedure continued to evaluate further corrections to w and R. However, we shall stop at this step.

6. Minimum Rayleigh number for convection

The value of R obtained by this procedure is the eigenvalue corresponding to the function w which, though oscillating, remains bounded in time. In general R is a function of the horizontal wave number α and the amplitude of the perturbation, ϵ. Thus

$$R(\alpha,\epsilon) = R_0(\alpha)+\epsilon^2R_2(\alpha)+\dots. \tag{37}$$

As a function of α there will be a least value R_c of R at say $\alpha = \alpha_c$. This critical value of α occurs when $\partial R/\partial\alpha = 0$, that is when

$$\partial R_0/\partial\alpha_c+\epsilon^2\,\partial R_2/\partial\alpha_c+\dots = 0. \tag{38}$$

Assume α_c is expanded in powers of ϵ,

$$\alpha_c = \alpha_0+\epsilon\alpha_1+\epsilon^2\alpha_2+\dots; \tag{39}$$

then (38) becomes

$$\partial R_0/\partial\alpha_0+\epsilon(\partial^2R_0/\partial\alpha_0^2)\,\alpha_1+\epsilon^2[\tfrac{1}{2}(\partial^3R_0/\partial\alpha_0^3)\,\alpha_1^2+(\partial^2R_0/\partial\alpha_0^2)\,\alpha_2+\partial R_2/\partial\alpha_0]\dots = 0,$$

so that

$$\left.\begin{aligned}\partial R_0/\partial\alpha_0 &= 0,\\ \alpha_1 &= 0,\\ \alpha_2 &= -(\partial R_2/\partial\alpha_0)/(\partial^2R_0/\partial\alpha_0^2).\end{aligned}\right\} \tag{40}$$

The first of these expressions gives $\alpha_0^2 = \frac{1}{2}\pi$. A similar expansion obtains R_c:

$$\begin{aligned}R_c(\epsilon) &= R_{0c}+\epsilon^2R_{2c}+\epsilon^4R_{4c}+\dots\\ &= R(\alpha_c,\epsilon)\\ &= R_0(\alpha_0)+\epsilon(\partial R_1/\partial\alpha_0)\,\alpha_1+\epsilon^2[\tfrac{1}{2}(\partial^2R_0/\partial\alpha_0^2)\,\alpha_1^2+(\partial R_0/\partial\alpha_0)\,\alpha_2+R_2(\alpha_0)]+\dots\\ &= R_0(\alpha_0)+\epsilon^2R_2(\alpha_0)+\dots\end{aligned} \tag{41}$$

in view of (40). Thus, to order ϵ^2, R_c is determined by evaluating R_0 and R_2 at $\alpha = \alpha_0$. It is only when one reaches R_4 that α_2 must be taken into account. In the next section, the values of R_{2c} are found for three particular cases.

7. Results

The values of R_{2c} will be obtained for the following cases: (*a*) when the oscillating temperature field is symmetric, i.e. the plate temperatures are modulated in phase, so $\phi = 0$; (*b*) when the field is antisymmetric, corresponding to an out-of-phase modulation, $\phi = \pi$; and (*c*) when only the temperature of the bottom plate is modulated, the upper plate being held at a fixed constant temperature. This case can be recovered from the equations by setting $\phi = -i\infty$.†

In all three cases the expression for $B_n(\lambda)$ simplifies considerably. Let

$$b_n = -\frac{4\pi^2 n\lambda^2}{[\lambda^2+(n+1)^2\pi^2][\lambda^2+(n-1)^2\pi^2]}, \tag{42}$$

then, for case (*a*)

$$\begin{aligned} B_n &= b_n \quad \text{if } n \text{ is even,} \\ &= 0 \quad \text{if } n \text{ is odd;} \end{aligned}$$

for case (*b*)

$$\begin{aligned} B_n &= 0 \quad \text{if } n \text{ is even,} \\ &= b_n \quad \text{if } n \text{ is odd;} \end{aligned}$$

and for case (*c*) $B_n = -b_n$ for all n (see footnote on this page).

The variable λ was defined in (9), which in terms of the dimensionless frequency reduces to

$$\lambda = (1-i)(\omega/2)^{\frac{1}{2}},$$

and thus

$$|b_n|^2 = \frac{16\pi^4 n^2\omega^2}{[\omega^2+(n+1)^4\pi^4][\omega^2+(n-1)^4\pi^4]}. \tag{43}$$

We also need an expression for

$$C_n = [L(\omega,n)+L^*(\omega,n)]/2|L(\omega,n)|^2$$

evaluated at $\alpha^2 = \frac{1}{2}\pi$. This reduces to

$$C_n = \frac{(\omega^2/\sigma)(n^2+\frac{1}{2})\pi^2-(n^2+\frac{1}{2})^3\pi^6+\frac{27}{8}\pi^6}{[(\omega^2/\sigma)(n^2+\frac{1}{2})\pi^2-(n^2+\frac{1}{2})^3\pi^6+\frac{27}{8}\pi^6]^2+\omega^2(1+(1/\sigma))^2(n^2+\frac{1}{2})^4\pi^8}, \tag{44}$$

and finally

$$R_{2c} = \tfrac{729}{64}\pi^{10}\Sigma|b_n|^2 C_n, \tag{45}$$

where the sum extends over even values of n for case (*a*), odd values for case (*b*) and all values for case (*c*). The series defined by (45) converges rapidly since the terms decrease as $1/n^{12}$.

Numerical results of R_{2c} as a function of ω for various values of σ are exhibited in the accompanying figures.

8. Limiting cases

Some features of the behaviour of R_{2c} as a function of ω can be seen by examining the limiting cases for very small or very large values of ω. When ω is very small,

$$C_1|b_1|^2 \to 1/[3\sigma(1+1/\sigma)^2\pi^6/2],$$

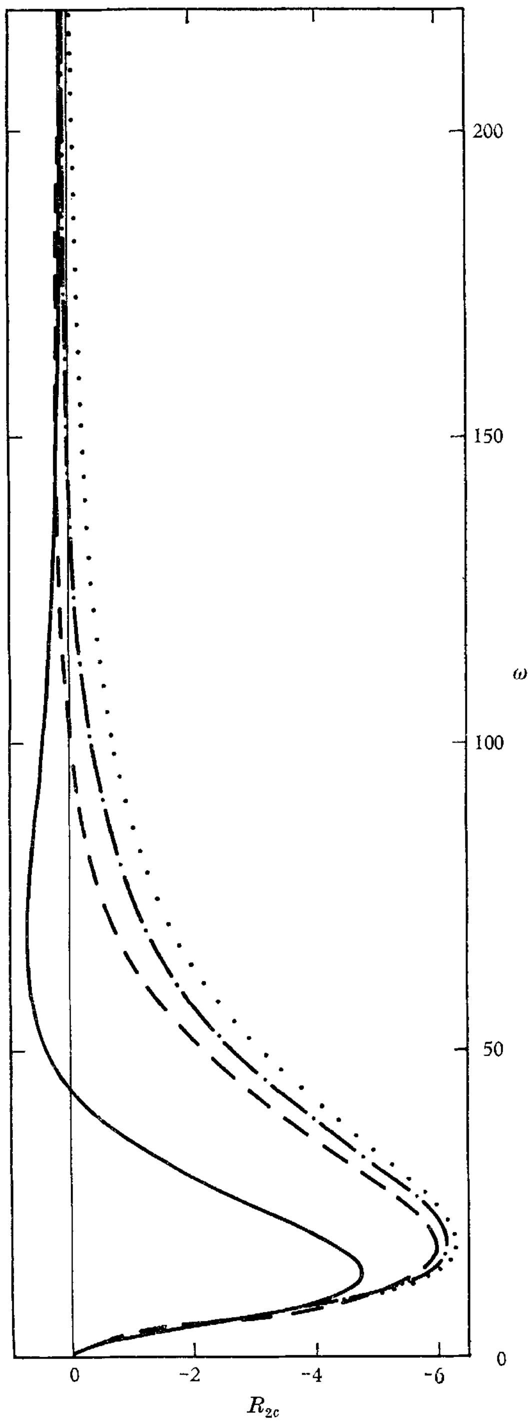
while for $n \neq 1$

$$C_n|b_n|^2 \to -\frac{16n^2\omega^2}{(n^2-1)^5(n^4+\frac{5}{2}n^2+\frac{13}{4})\pi^{10}},$$

so the general form of R_{2c} near $\omega = 0$ is

$$R_{2c} \to R_\sigma - \beta\omega^2,$$

† In this case it is convenient to take the wall temperature to be $T_R+\Delta T/2+\epsilon\Delta T\cos\omega t$ at the bottom and $T_R-\Delta T/2$ at the top.



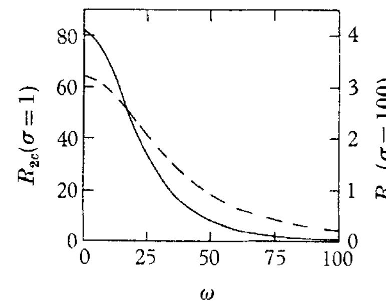


FIGURE 2. R_{2c} as a function of ω when the wall temperatures are modulated out of phase, for: ——, $\sigma=1$; – –, $\sigma=100$. Note the different vertical scales for the two cases.

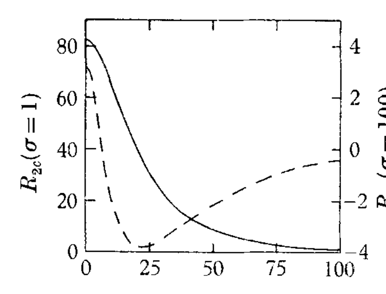


FIGURE 3. R_{2c} as a function of ω when only the temperature of the lower wall is modulated, for: ——, $\sigma=1$; – –, $\sigma=100$. Note the different vertical scales for the two cases.

FIGURE 1. R_{2c} as a function of ω when the wall temperatures are modulated in phase, for: ——, $\sigma=1$; —, $\sigma=5$; —·—, $\sigma=10$; · · · ·, $\sigma=1000$.

where $$R_\sigma = 27\pi^4/8\sigma(1+1/\sigma)^2, \tag{46}$$

and β is a constant, which depends only on the case being considered.

In the case of symmetric excitation, the sum extends only over even values, so that

$$R_{2c} \to -0\cdot102\omega^2,$$

indicating that the dependence on σ can only appear at reasonably large values of ω. The effect of modulation in this case is to destabilize the system; with convection occurring at an earlier point than in the unmodulated system. This agrees with the results of Krishnamurti (1967) in her analysis of convection with a slowly varying mean temperature, which corresponds to low-frequency symmetric excitation.

In the antisymmetric case

$$R_{2c} \to R_\sigma - 0\cdot0005\omega^2,$$

so the effect is one of stabilization, decreasing with frequency. The maximum value of R_σ obtains at $\sigma = 1$ and is 82·1. Since R_{0c} is only eight times as large, there is a good chance that this effect can be observed experimentally, for a moderately large amplitude of modulation, assuming that, at least qualitatively, these results can be extrapolated to ϵ near 1.

The value of R_{2c} for the case in which only the bottom temperature is modulated is obtained by adding the other two, so that

$$R_{2c} \to R_\sigma - 0\cdot103\omega^2.$$

For σ near 1, this is not significantly different from case (*b*); however, for larger σ, R_σ can become sufficiently small to be overtaken by the other terms in the sum.

As ω tends to infinity, R_{2c} tends to zero as $1/\omega^2$, so the effect of modulation disappears altogether. This agrees with Donnelly's experiments on the stability of Taylor vortices. For intermediate values of ω, the effect of changing the frequency makes itself evident in the numerator of C_n. If ω^2/σ is as large as

$$(n^2+\tfrac{1}{2})^2\pi^4$$

C_n is positive rather than negative. Indeed, C_2 is zero when

$$\omega = \pi^2(78\sigma)^{\frac{1}{2}}/2, \tag{47}$$

so that in the symmetric case R_{2c} should be zero near that value of ω, a prediction which is borne out by the numerical calculations. Thus, for example, for $\sigma = 10$, (47) gives $\omega = 138$, while, from the numerical results, R_{2c} is zero at $\omega = 145$. The peak negative value of R_{2c} is more difficult to estimate, but appears from the numerical evaluation of the series that it occurs near $\omega = 20$ and has a value of about -6, over the entire range of σ.

None of the cases considered duplicates the behaviour observed by Donnelly in his experiments, in which a peak stabilization occurs at a value of ω different from zero. This is probably due to the fact that, while the two problems are fairly similar, they are not identical.

9. Discussion

The analysis presented in this paper is based on the assumptions that the amplitude of the modulating temperature is small compared with the imposed steady temperature difference, and that the convective currents are weak so that non-linear effects may be neglected. The violation of these assumptions would alter the results significantly only when the modulating frequency is low. At high frequencies, the hydrostatic temperature field given by (6)–(8) has a boundary-layer character so that it has essentially a linear gradient, except near the boundary, in regions of dimensionless thickness $(1/\omega)^{\frac{1}{2}}$. Since it is known that for a linear gradient instability cannot occur at Rayleigh numbers lower than the minimum value given by the linearized theory, it can be expected that, for $\omega \gg 1$, modulation would have little effect on the onset of convection, even when the amplitude of the modulation is large.

When the frequency of modulation is low, the effect of modulation on the temperature field is felt throughout the fluid layer. If the plates are modulated in phase, the temperature profile consists of the steady straight line section plus a parabolic profile which oscillates in time. As the amplitude of the modulation increases, the parabolic part of the profile becomes more and more significant. It is known that a parabolic profile is subject to finite-amplitude instabilities so that convection occurs at lower Rayleigh numbers than those predicted by the linear theory. This was discussed theoretically by Veronis (1963) for a layer of water in which a parabolic density profile is produced by a temperature field with temperatures near 4 °C, and experimentally by Krishnamurti, who obtained parabolic profiles by increasing or decreasing the mean temperature linearly in time.

Since the amplitude of the modulation is an externally controlled variable, these finite-amplitude instabilities can be avoided by simply not allowing the amplitude to become very large. The amplitude of the convection currents, however, cannot be controlled, but rather is determined by the non-linear interactions. It is important that the flow fields considered remain of small amplitude throughout a cycle of modulation, otherwise the assumption that the non-linear terms are small is violated. This can occur for low-frequency modulation in the cases when the plate temperatures are oscillated out of phase or when only one of the plate temperatures is modulated. In these cases, the temperature field has essentially a linear gradient varying in time, so that the instantaneous Rayleigh number is supercritical for half a cycle and subcritical during the other half cycle. The amplitude of the modulation ϵ gives the maximum fractional departure of the Rayleigh number from its critical value. An estimate of the growth of the convective flow during the supercritical half cycle can be obtained from the linearized theory, which predicts an exponential growth with a time constant of the order of $R_c/\Delta R = 1/\epsilon$. Thus, if the period of oscillation is short compared with $1/\epsilon$, the amplitude of the convective flow remains small, and the non-linear terms may be neglected. This requirement restricts the validity of the results given here to frequencies such that $\omega > \epsilon$. The same conclusion is reached by imposing the condition that the amplitude of ϵw_1 should not exceed that of

w_0. w_1 is given in (32), and it can be seen from that expression that w_1 becomes large if $L(\omega, n)$ is small. This happens if $n = 1$ and ω is small, since $L(\omega, 1)$ is proportional to ω. In that case w_1 is of order $1/\omega$, and hence, if $|\epsilon w_1| < |w_0|$, we must have $\omega > \epsilon$.

The author is grateful to Professors A. Ingersoll and W. V. R. Malkus for their valuable advice and interest in this study.

This work was supported by the Office of Naval Research.

REFERENCES

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Donnelly, R. J. 1964 *Proc. Roy. Soc.* A **281**, 130.

Ingersoll, A. 1966 *Phys. Fluids*, **9**, 682.

Krishnamurti, R. E. 1967 Ph.D. Dissertation, University of California, Los Angeles.

Malkus, W. V. R. & Veronis, G. 1958 *J. Fluid Mech.* **4**, 225.

Schlüter, A., Lortz, D. & Busse, F. 1965 *J. Fluid Mech.* **23**, 129.

Veronis, G. 1963 *Astrophys. J.* **137**, 641.